Ray and Eikonal Theory I

→ Rays, Eikonal Theory and Wave Propagation.

eikonal → icon
(Greek) ↓
image

OV:

→ here, seek to provide description of wave propagation in 'short wavelength' limit [N.B. How short? - see HW on parabolic wave equation].
→ relevant to semi-classical limit of QM
→ description is in terms of rays - paths followed by wave.

Now:

- from previous HW, Fermat's minimum time principle (1662)

i.e. $T = \int_1^2 \frac{ds}{c(\underline{x})} = \frac{1}{c_0}\int_1^2 ds\, n(\underline{x})$

travel time

index

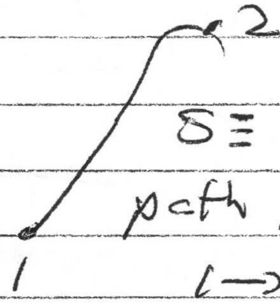
'ray Lagrangian'

$\delta T = 0 \Rightarrow$ ray path.

Generalizing the HW:

Fermat $\Rightarrow$

$$0 = \delta \int_1^2 n(\underline{x}(s))\, ds$$



$s \equiv$ ray path parameter

$1 \longrightarrow 2$

$$= \delta \int_1^2 n(\underline{x}(s)) \left(\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} \right)^{1/2} ds$$ (dummy time)

$$\equiv \text{[redacted]} \int_1^2 L\, ds$$

$\Rightarrow$

$$0 = \int_1^2 \left(\frac{\partial L}{\partial \underline{x}} \cdot \delta \underline{x} + \frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds} \right)} \cdot \delta \left(\frac{d\underline{x}}{ds} \right) \right)$$

$$\underset{\text{chp}}{=} \text{e.p.} + \int_1^2 \left(\frac{\partial L}{\partial \underline{x}} \cdot \delta \underline{x} - \frac{d}{ds} \left(\frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds} \right)} \right) \cancel{\delta \underline{x}} \right)$$

$\Rightarrow$

$$\boxed{\begin{gathered} \frac{\partial L}{\partial \underline{x}} - \frac{d}{ds} \left(\frac{\partial L}{\partial \left(\frac{d\underline{x}}{ds} \right)} \right) = 0 \\ L = n(\underline{x}(s)) \left(\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds} \right)^{1/2} \end{gathered}}$$

Crank $\Rightarrow$

if $|\dot{\underline{x}}| = \left[\frac{d\underline{x}}{ds} \cdot \frac{d\underline{x}}{ds}\right]^{1/2}$

$$\boxed{|\dot{\underline{x}}| \frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n(\underline{x}) \frac{\dot{\underline{x}}}{|\dot{\underline{x}}|} \right) = 0}$$

→ general expression

→ $\partial n / \partial \underline{x}$ ⟺ effective force on ray
($U \leftrightarrow n$)

→ $n(\underline{x}) \frac{\dot{\underline{x}}}{|\dot{\underline{x}}|}$ ⟺ defines generalized momentum analogue.
$\left(n(\underline{x}) \frac{d\underline{x}}{ds} \right)$

Note: $ds^2 = d\underline{x} \cdot d\underline{x}$
so $|\dot{x}| = 1$

$$\Rightarrow \boxed{\frac{\partial n}{\partial \underline{x}} - \frac{d}{ds}\left(n(\underline{x}) \frac{d\underline{x}}{ds} \right) = 0}$$

is equivalent.

→ A bit of geometry:

$$\frac{d}{ds}\left(n(\underline{x}) \frac{d\underline{x}}{ds} \right) - \frac{\partial n}{\partial \underline{x}} = 0$$ → ray equation

⇒

$$n(\underline{x}) \frac{d^2\underline{x}}{ds^2} + \left(\frac{\partial n}{\partial \underline{x}} \cdot \frac{d\underline{x}}{ds} \right) \frac{d\underline{x}}{ds} = \frac{\partial n}{\partial \underline{x}}(\underline{x})$$

∴

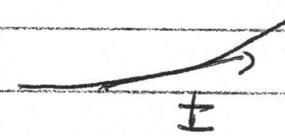
$$\boxed{\frac{d^2\underline{x}}{ds^2} = \frac{1}{n(\underline{x})} \frac{\partial n}{\partial \underline{x}} - \frac{1}{n(\underline{x})} \left(\frac{\partial n}{\partial \underline{x}} \cdot \frac{d\underline{x}}{ds} \right)}$$

What does it mean?

→ $d\underline{x}/ds$ is unit tangent to ray.

i.e. $ds\,ds = d\underline{x} \cdot d\underline{x}$

$\underline{t} = d\underline{x}/ds$



so

→ $d^2\underline{x}/ds^2$ corresponds to ray curvature κ.

$1/|\underline{K}| \equiv$ effective radius of curvature

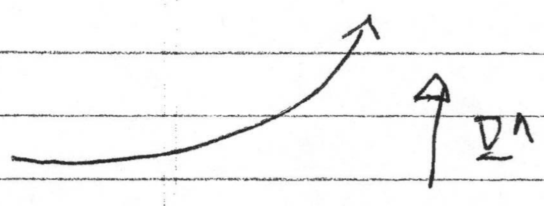
so

$$\underline{K} = \frac{1}{n}\underline{\nabla} n - \frac{1}{n}(\underline{t}\cdot\underline{\nabla} n)\,\underline{t}$$

$$= \frac{1}{n}(\underline{\nabla} n \cdot \underline{\hat{n}}_0)\,\underline{\hat{n}}_0$$

unit normal to path

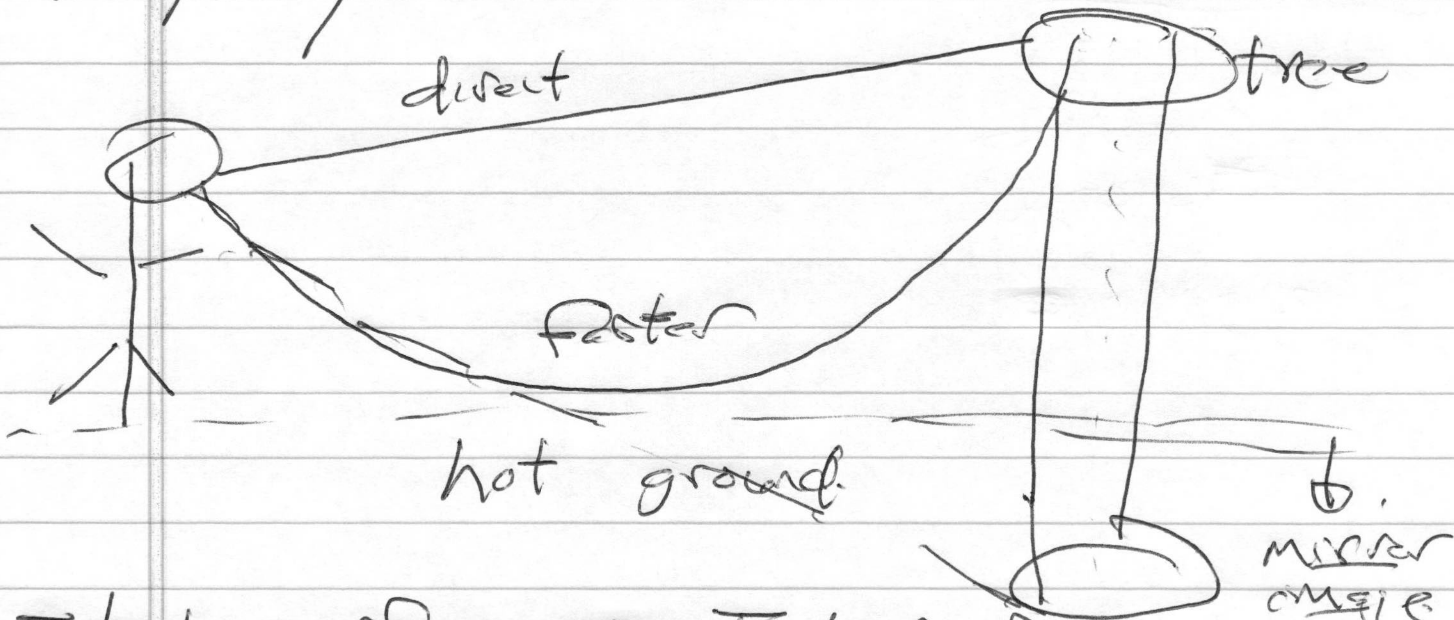
Loosely put, ray curves toward region of increasing index.



→ Mirages

- mirages are optical illusions of reflection from water, etc. which occur in deserts.

- how / why



- hot surface ⇒ T decreases
air density increases with height

- index n ~ density

- so, observer sees direct path image and curved path-induced mirror image

- if no tree → blue sky → appears like water → mirage

- so reasonable to take index ~ Z

$n(z) = n_0 (1 + \alpha z)$

Now, Fermat $\Rightarrow$ ray from:

$$\delta \int (1 + (dz/dx)^2)^{1/2} n(z) = 0$$

So

$$\frac{d}{dx}\left(\frac{n(z)}{\left(1+(dz/dx)^2\right)^{1/2}}\frac{dz}{dx}\right) = \left(1+\left(\frac{dz}{dx}\right)^2\right)^{1/2}\frac{dn}{dz}$$

$dz/dx \equiv \dot{z}$

$\Rightarrow$

$$\frac{d}{dx}\left(\frac{n_0(1+\alpha z)}{(1+\dot{z}^2)^{1/2}}\,\dot{z}\right) = n_0(1+\dot{z}^2)^{1/2}\,\alpha$$

For ~~[illegible]~~ $\sim$ horizontal rays,

$\dot{z}^2 \ll 1$

$\alpha z \ll 1$

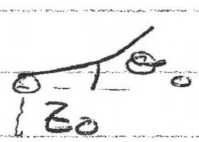
$\Rightarrow$

$$\frac{d^2 z}{dx^2} \cong \alpha$$

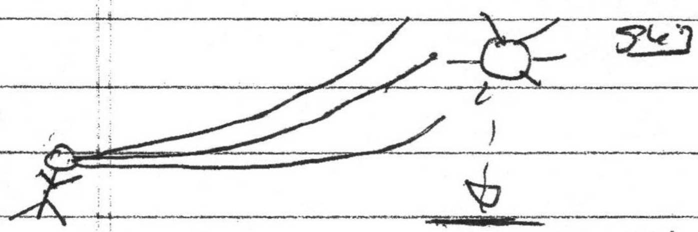
$\therefore$ then have:

$$z(x) = \left(\frac{\alpha}{2}x^2 + \tan\theta_0\, x + z_0\right)$$

inclination



then rays diverge parabolically,



apparent location
(shimmering, bright light)

⇒ mirage
(appears like reflection
from water)

Origin of shimmer? ⇒ conv. turbulence

Now, consider:

→ Helmholtz Eqn.

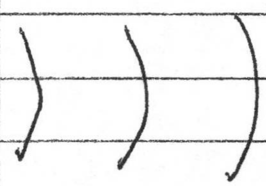
$$\nabla^2 \psi + \frac{\omega^2}{c(x)^2}\psi = 0$$

→ index

$$1/c(\underline{x})^2 \equiv \frac{n(x)^2}{c_0^2}$$

c_0^2 → ref. speed.

→ consider phase front

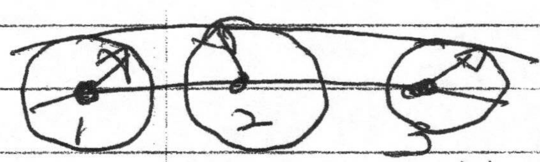


ϕ = const. surfaces

↓

phase.

Now, to describe propagation:



ϕ = const. surf. at $t+\Delta t$

ϕ = const surface at t_0

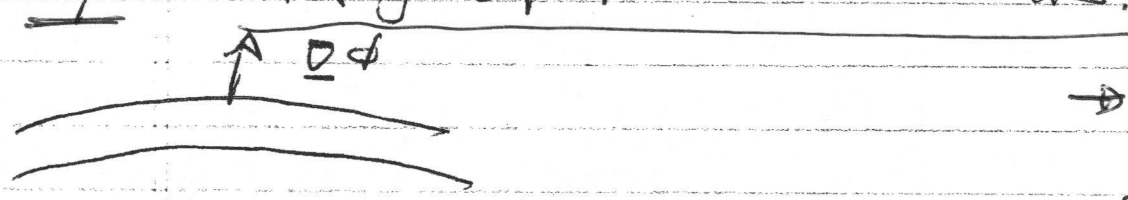
$c_1 \Delta t$ $c_2 \Delta t$ $c_3 \Delta t$

i.e. each point on surface ϕ = const at t emits spherical disturbance.

- sum of spherical disturbances defines new constant phase surface. Curvature due $c(\underline{x})$.

  Envelope of spheres $\Rightarrow$ wavefront at $t+\Delta t$

- rays orthogonal to wave fronts.



$\Rightarrow$ root motivation of Hamiltonian Mech.

Now, $\forall$ infinitesimal displacement vector along ray $\equiv d\underline{\sigma}$

i.e. $d\underline{\sigma} \parallel \underline{\nabla}\phi$

then, since equivalent to advance in space or time,

$$\underline{\nabla}\phi \cdot d\underline{\sigma} = \omega\, dt$$

$$|\underline{\nabla}\phi|\,|d\sigma| = \omega\, dt$$

$$dt = d\sigma/c \qquad \text{(by definition)}$$

$$\Rightarrow \quad |\underline{\nabla}\phi|\,|d\sigma| = \omega \frac{d\sigma}{c}$$

$\therefore$

$$|\underline{\nabla}\phi| = \omega/c$$

$$\Rightarrow \boxed{(\underline{\nabla}\phi)^2 = \omega^2/c^2}$$

– eikonal equation

⇒ eqn. for spatial evolution ϕ.

reduces wave eqn to phase eqn.

N.B. – Can obtain directly from Helmholtz Eqn.

$$\nabla^2\psi + \frac{\omega^2}{c(x)^2}\psi = 0$$

$$\psi = A\, e^{i\phi(\underline{x})/\epsilon}$$

$\epsilon \to 0$ (short wavelength)

(WKB)

⇒

$$\left[-\frac{(\nabla\phi)^2}{\epsilon^2}A + i\frac{\nabla^2\phi}{\epsilon}A + 2i\frac{\underline{\nabla}A\cdot\underline{\nabla}\phi}{\epsilon}\right.$$

$$\left. + \nabla^2 A\right] e^{i\phi} = -\omega^2/c(x)^2\, A\, e^{i\phi}$$

so dominant balance

$$+\frac{(\nabla\phi)^2}{\epsilon^2\ \text{[redacted]}} = \frac{\omega^2}{c(\underline{x})^2}$$

- now absorb ϵ to ϕ.

- note eikonal lowers order of problem $\Rightarrow$ first order pde.

Now, by construction

$\underline{\nabla}\phi \cdot d\underline{\ell} \equiv$ net phase increment along ray.

so $\underline{\nabla}\phi = \underline{k} = \underline{k}(\underline{x})$

$\uparrow$ in sense of WKB

(n.b. generally, $\partial\phi/\partial t = -\omega$).

$$\phi \doteq \int \underline{k}\cdot d\underline{x} = \int \underline{\nabla}\phi \cdot d\underline{x}$$

$$\equiv \int \underline{k}\cdot d\underline{\ell}$$

$$\psi = A \exp\left[i\left(\int \underline{k}\cdot d\underline{x} - \omega t\right)\right]$$

is eikonal approximation to wave fctn.

N.B. $\Rightarrow$ $\underline{k}$ specifies ray direction
orthog. to ϕ (phase) surfaces.

$\Rightarrow$ Now, seek equations which evolve ray path in time, space i.e.
give - ray position $\underline{x}$ as fctn of
- ray direction $\underline{k}$
time.
$\Rightarrow$ defines mechanics problem.

a) Poor Man's Version

- For linear wave, have $\omega = \text{const.}$

Since $\omega = \omega(\underline{k}, \underline{x}) \Rightarrow$

$$\frac{d\omega}{dt} = 0 = \cancel{\frac{\partial \omega}{\partial t}} + \frac{\partial \omega}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt}$$

$$\Rightarrow \boxed{\begin{aligned} d\underline{k}/dt &= -\partial\omega/\partial\underline{x} \\ d\underline{x}/dt &= \partial\omega/\partial\underline{k} = \underline{V}_{gr} \end{aligned}}$$

eikonal equations

Hamiltonian EOMs

with, of course:

$$\omega^3 = c(\underline{x})^2 k^2$$

$$\cancel{2}\omega\,\partial\omega = \cancel{2}\underline{k}\cdot\partial\underline{k}\;\; c(\underline{x})^2$$

$$\underline{k} = k\,\hat{k}$$

$$\hat{k} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\partial\omega = \hat{k}\cdot\partial\underline{k}\;\; c(x)$$

$$d\omega/\partial\underline{k} = c(\underline{x})\,\hat{k}$$

$\equiv$ group velocity.

$$\frac{\partial\omega}{\partial\underline{x}} = \frac{\partial}{\partial\underline{x}}\left[c(x)^2 k^2\right]^{1/2} = k\,\frac{\partial c(\underline{x})}{\partial\underline{x}}$$

$\Rightarrow$

$$\boxed{\begin{aligned}\frac{d\underline{x}}{dt} &= c(\underline{x})\,\hat{k}\\ \frac{d\underline{k}}{dt} &= -k\,\frac{\partial c(\underline{x})}{\partial\underline{x}}\end{aligned}}$$

$c(\underline{x})$ profile determines ray path.

eikonal equation for acoustics.

b.) More Rigorously ...

$$\Phi = \int\left[\underline{k}\cdot d\underline{x} - \omega\,dt\right]$$ $\rightarrow$ total phase

$$dS = L\,dt$$
$$= (p\dot{q} - H)\,dt$$

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\,dt = (\underline{k}\cdot\dot{\underline{x}} - \omega)\,dt$$

Now, assert ray will follow path which extremizes Φ, i.e. { minimizes accumulated phase }.

Note: analogy of phase and action.

: later demonstrate connection to Fermat.

$$\delta\Phi = \delta\int\left[\underline{k}\cdot d\underline{x} - \omega\,dt\right] = 0$$

$$= \int\left\{\left[\delta\underline{k}\cdot d\underline{x} + \underline{k}\cdot\delta d\underline{x}\right] - \left(\frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k} + \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x}\right)dt\right\}$$

as usual, $\delta\underline{x} = \delta\underline{k} = 0$, at end points.

So integrating by parts:

$$\delta\Phi = \int\left\{\left[\delta\underline{k}\cdot d\underline{x} - d\underline{k}\cdot\delta\underline{x}\right] - \left[\left(\frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right) + \left(\frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x}\right)\right]dt\right\} + \text{e.p.}$$

$\Rightarrow$ $$d\underline{x} = \left(\frac{\partial \omega}{\partial \underline{k}}\right) dt$$

$$d\underline{k} = -\left(\frac{\partial \omega}{\partial \underline{x}}\right) dt$$

$\Rightarrow$ $$\boxed{\begin{aligned} \frac{d\underline{x}}{dt} &= \frac{\partial \omega}{\partial \underline{k}} \\ \frac{d\underline{k}}{dt} &= -\frac{\partial \omega}{\partial \underline{x}} \end{aligned}}$$ $\rightarrow$ eikonal equations

Note:

$\rightarrow$ evolve ray in $\underline{x}$, $\underline{k}$ phase space.

$\underline{x}$ $\downarrow$ position; $\underline{k}$ $\searrow$ direction momentum

$\rightarrow$ Hamiltonian equations for ray in

$(\underline{x}, \underline{k})$ phase space.

i.e. $$\frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} + \frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt}$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial \omega}{\partial \underline{k}} - \frac{\partial}{\partial \underline{k}} \cdot \frac{\partial \omega}{\partial \underline{x}} = 0.$$

→ since eikonal equations Hamiltonian, can define:

$\rho(\underline{x}, \underline{k}, t)$ ≡ wave density in $\underline{x}, \underline{k}$ phase space.

$\downarrow$

$N(\underline{x}, \underline{k}, t)$
- wave action density
- ~ Wigner dist.
- ~ intensity.

and use Liouville's Thm:

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{v}_{gr} \cdot \frac{\partial \rho}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0}$$

- wave kinetic eqn.
- relates ρ, and intensity, to $c(\underline{x})$ profiles, for acoustics
- gives intensity evol'n.
- applications in radiation hydro, quasi-particle evolution, etc.

→ Obvious analogy: (Hamiltonian systems)

Particles	Rays
H	ω
$\underline{p}$	$\underline{k}$
$\underline{q}$	$\underline{x}$
S	ϕ

→ Eikonal Theory; Supplement

Recall:

— for Helmholtz Eqn, derived:

$$(\underline{\nabla}\phi)^2 = \frac{\omega^2}{c(x)^2} \quad \Rightarrow \text{ eikonal eqn.}$$

↳ inhomogeneous speed.

$$\psi \sim A e^{i\phi}$$

— as rays ⊥ phase fronts

$$\underline{\nabla}\phi \equiv \underline{k} = k(\underline{x})$$

à la WKB

$$\omega = -\frac{\partial\phi}{\partial t}$$

so

$$\psi = A \exp\left[i \overbrace{\left(\int \underline{k}(\underline{x}) \cdot d\underline{x} - \omega t \right)}^{\Phi} \right]$$

eikonal approx. to wave function

— now, for ray trajectories, observe total phase Φ

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\, dt$$

$$= \left(\underline{k}\cdot \frac{d\underline{x}}{dt} - \omega\right) dt$$

$S \leftrightarrow \Phi$ is key analogy

analogous

$$S = \int L\, dt \quad \Rightarrow \quad dS = L\, dt$$
$$= (\underline{p}\dot{\underline{q}} - H)\, dt$$

$\therefore$

— obvious analogy

$\underline{k} \Leftrightarrow \underline{p}$ i.e. QM: $\underline{p} = \hbar \underline{k}$

$\underline{x} \leftrightarrow \underline{q}$ $E = \hbar\omega$

$\omega \leftrightarrow H$

so $\dfrac{d\underline{k}}{dt} = -\dfrac{\partial \omega}{\partial \underline{x}} \Leftrightarrow \dfrac{d\underline{p}}{dt} = -\dfrac{\partial H}{\partial \underline{q}}$

$\frac{d\underline{p}}{dt} = -\partial H/\partial \underline{q}$

$\dfrac{d\underline{x}}{dt} = \dfrac{\partial \omega}{\partial \underline{k}} \Leftrightarrow \dfrac{d\underline{q}}{dt} = \dfrac{\partial H}{\partial \underline{p}}$

$\frac{d\underline{q}}{dt} = \frac{\partial H}{\partial \underline{p}}$

eg in terms $c(\underline{x})$:

$$\omega^2 = c(x)^2 k^2$$

$$\frac{d\underline{k}}{dt} = -k \, \frac{\partial c(\underline{x})}{\partial \underline{x}}$$

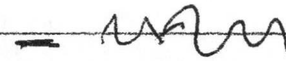
$$\frac{d\underline{x}}{dt} = c(\underline{x}) \, \hat{k} .$$

N.B.

→ $\partial\omega/\partial\underline{k} \equiv \underline{v}_{gr}$ group velocity

What does $\underline{v}_{gr}$ mean?

Consider wave packet,



carrier $\underline{k}_0$

spread $\Delta\underline{k}$

$$\phi\Big|_{t=0} \sim \underbrace{e^{i\underline{k}_0\cdot\underline{x}}}_{\text{carrier}} \; \underbrace{F(\underline{x})}_{\text{envelope}}$$

$$F(\underline{x}) \sim \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}}$$

so

$$\phi(\underline{x},t) \sim \sum_{\Delta\underline{k}} \underbrace{e^{i[(\underline{k}_0+\Delta\underline{k})\cdot\underline{x} - \omega(\underline{k}_0+\Delta\underline{k})t]}}_{\text{carrier}}$$

$$\sim e^{i(\underline{k}_0\cdot\underline{x} - \omega(\underline{k}_0)t)} \sum_{\Delta\underline{k}} e^{i\Delta\underline{k}\cdot\underline{x}} \, e^{-\frac{\partial\omega}{\partial\underline{k}}\cdot\Delta\underline{k}\,t}$$

$\underline{\infty}$ $\quad \phi(\underline{x},t) \sim e^{i(\underline{k}_0\cdot\underline{x}-\omega t)} F\Big(\underbrace{\underline{x} - \frac{\partial\omega}{\partial \underline{k}}}_{\underline{v}_{gr}} t\Big)$

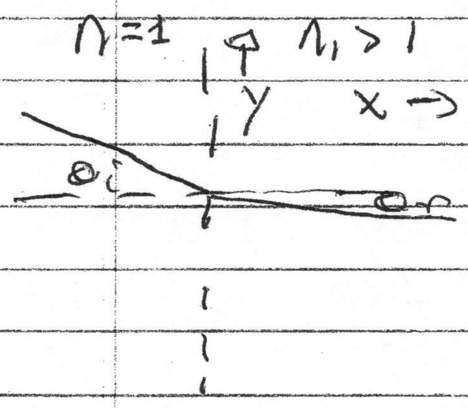
$\Rightarrow$ rate/speed at which energy propagated. $\rightarrow |\phi|^2$

N.B. $\quad E_{pckt} \sim |\phi_k|^2 \sim |f|^2$

$\Rightarrow$ v_{gr} sets speed at which energy propagates.

$\rightarrow \quad \frac{d\underline{k}}{dt} = -\frac{\partial\omega}{\partial \underline{x}} \quad \Rightarrow$ Snell's Law.

i.e.



$\frac{d\underline{k}}{dt} = \frac{-\partial\omega}{\partial \underline{x}} \quad \Rightarrow \quad \frac{dk_y}{dt} = 0$

$k_{y-} = k_{y+} \quad \Rightarrow \quad k_- \sin\theta_i = k_+ \sin\theta_r$

$k_-^2 = n_0^2 \frac{\omega^2}{c_0^2} \qquad k_+^2 = n_1^2 \omega^2 / c_0^2$

$$n_0 \sin\theta_i = n_1 \sin\theta_r \quad \checkmark$$

- Now, if seek first principles approach

$\Rightarrow$ extremize Φ (i.e. look for phase stationarity)

stationarity $\to$ trajectories i.e. ray as particle (cf. full wave)

$$\delta\Phi = \delta\int[\underline{k}\cdot d\underline{x} - \omega dt]$$

$$= \delta\int[\underline{k}\cdot\dot{\underline{x}} - \omega]\,dt$$

($t \to iT$ steepest descents; $\int d\underline{k}\, e^{i\Phi} \to$ packet)

$$= \int\left[\delta\underline{k}\cdot\dot{\underline{x}} + \underline{k}\cdot\delta\dot{\underline{x}} - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]dt$$

but $\delta\dot{\underline{x}} = \frac{d}{dt}\delta\underline{x}$, e.p. fixed.

$$\delta\Phi = \cancel{\underline{k}\cdot\delta\underline{x}\Big|_{t_1}^{t_2}} + \int\left[\delta\underline{k}\cdot\dot{\underline{x}} - \frac{d\underline{k}}{dt}\cdot\delta\underline{x}\right.$$

$$\left. - \frac{\partial\omega}{\partial\underline{x}}\cdot\delta\underline{x} - \frac{\partial\omega}{\partial\underline{k}}\cdot\delta\underline{k}\right]$$

$\delta \Phi \Rightarrow$

$$\frac{dx}{dt} = \frac{\partial \omega}{\partial k} , \qquad \frac{dk}{dt} = -\frac{\partial \omega}{\partial x}.$$

$\Rightarrow$ Liouville Thm $\rightarrow$ Wave Kinetics.

N.B.: for semi-classical limit

$$P = N\hbar \underline{k}$$
$$E = N\hbar\omega$$

$N \Leftrightarrow \rho$.

Finally, : to recover Fermat, note:

$$\delta \Phi = 0$$

$$d\phi = \underline{k} \cdot d\underline{x} - \omega t$$

so for ray path:

$$\delta \Phi_s = 0, \qquad \Phi_s = \int \underline{k} \cdot d\underline{x}$$

but

$$\underline{k} \cdot d\underline{x} = \underline{k} \cdot \frac{d\underline{x}}{ds}\, ds \Rightarrow$$

$$\underline{k} = \underline{\nabla}\phi = |\underline{\nabla}\phi|\, \underline{t}$$

$$d\underline{x}/ds = \underline{t}$$

so

$$\delta \Phi_s = \delta \int |\nabla\phi|\, ds$$

but $|\nabla\phi|^2 = \omega^2/c^2 = \frac{\omega^2}{c_0^2}\, n(\underline{x})^2$

$\Rightarrow$

$$\delta \Phi_s = \delta \frac{\omega}{c_0} \int ds\, n(\underline{x}) = 0$$

$$\Rightarrow \delta \int ds\, n(\underline{x}) = 0$$